9 Fixed Effects

```r
library(plm) # estimating panel models
library(lmtest) # regression inference
library(stargazer) # regression outputs
```

9.1 Time-constant Variables

Panel data allows us to control for variables that are constant over time, even if these variables are not directly observable.

Consider a basic panel regression model:

$$Y_{it} = \beta_1 + \beta_2 X_{it} + \beta_3 Z_i + u_{it}. \tag{9.1}$$

Here, Z_i represents a variable that does not change over time and is specific to an individual (e.g., gender, ethnicity, parental education).

For simplicity, assume here that observations are only available for two time periods ($t = 1$ and $t = 2$). We can focus on the changes between these periods.

Subtracting the right-hand side of Equation 9.1 at $t = 1$ from $t = 2$ gives

$$\begin{aligned} &\beta_1 + \beta_2 X_{i2} + \beta_3 Z_i + u_{i2} - (\beta_1 + \beta_2 X_{i1} + \beta_3 Z_i + u_{i1}) \\ &= \beta_2 \Delta X_{i2} + \Delta u_{i2}. \end{aligned}$$

The symbol Δ represents first-differencing, i.e. $\Delta X_{i2} = X_{i2} - X_{i1}$ and $\Delta u_{i2} = u_{i2} - u_{i1}$.

By **first-differencing** both sides of Equation 9.1, our model becomes

$$\Delta Y_{i2} = \beta_2 \Delta X_{i2} + \Delta u_{i2}. \tag{9.2}$$

β_1 and $\beta_3 Z_i$ do not appear in the transformed model Equation 9.2 because they are time-constant and cancel out.

In this differenced model, β_2 can be estimated by regressing ΔY_{i2} on ΔX_{i2} without an intercept. This regression isolates the marginal effect of X_{it} on Y_{it} conditional on any unobserved individual characteristics like Z_i. β_2 is the marginal effect of X_{it} on Y_{it} given the same individual-specific time-constant characteristics.

We can control for any time-constant variable without actually observing it. This is a remarkable advantage over conventional cross-sectional regression or pooled panel regression.

We may combine the terms β_1 and $\beta_3 Z_i$ and define the **individual-specific** effect $\alpha_i = \beta_1 + \beta_3 Z_i$. The term α_i is also called **individual fixed effect**. The fixed effect cancels out after taking first differences.

9.2 Fixed Effects Regression

Consider a panel dataset with dependent variable Y_{it}, a vector of k independent variables $\boldsymbol{X}_{it}$, and an individual fixed effect α_i for $i = 1, \dots, n$ and $t = 1, \dots, T$.

Because α_i already represents any time-constant variable of individual i, we assume that all variables in $\boldsymbol{X}_{it}$ are time-varying. That is, $\boldsymbol{X}_{it}$ neither contains an intercept nor any time-constant variables like gender, birthplace, etc.

Fixed-effects Regression

The fixed-effects regression model equation for individual $i = 1, \dots, n$ and time $t = 1, \dots, T$ is

$$Y_{it} = \alpha_i + \boldsymbol{X}_{it}'\boldsymbol{\beta} + u_{it}, \tag{9.3}$$

where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)'$ is the $k \times 1$ vector of regression coefficients and u_{it} is the error term for individual i at time t.

The fixed effects regression assumptions are:

- (A1-fe) **conditional mean independence**: $E[u_{it} | \boldsymbol{X}_{i1}, \dots, \boldsymbol{X}_{iT}, \alpha_i] = 0$.
- (A2-fe) **random sampling**: $(\alpha_i, Y_{i1}, \dots, Y_{iT}, \boldsymbol{X}_{i1}', \dots, \boldsymbol{X}_{iT}')$ are i.i.d. draws from their joint population distribution for $i = 1, \dots, n$.
- (A3-fe) **large outliers unlikely**: $0 < E[Y_{it}^4] < \infty$, $0 < E[u_{it}^4] < \infty$.
- (A4-fe) **no perfect multicollinearity**: $\boldsymbol{X}$ has full column rank.

9.3 Differenced Estimator

The first-differencing transformation can be used to estimate Equation 9.3:

$$\Delta Y_{it} = Y_{i,t} - Y_{i,t-1}, \quad \Delta \boldsymbol{X}_{it} = \boldsymbol{X}_{i,t} - \boldsymbol{X}_{i,t-1}.$$

Taking first differences on both sides of Equation 9.3 implies

$$\Delta Y_{it} = (\Delta \boldsymbol{X}_{it})' \boldsymbol{\beta} + \Delta u_{it}, \tag{9.4}$$

where $\Delta u_{it} = u_{i,t} - u_{i,t-1}$. Notice that the fixed effect α_i cancels out.

Hence, we can apply the OLS principle to Equation 9.4 to estimate $\boldsymbol{\beta}$. We regress the differenced dependent variable ΔY_{it} on the differenced regressors $\Delta \boldsymbol{X}_{it}$ for $i = 1, \ldots, n$ and $t = 2, \ldots, T$.

A problem with this differenced estimator is that the transformed error term Δu_{it} defines an artificial correlation structure, which makes the estimator non-optimal. $\Delta u_{i,t+1} = u_{i,t+1} - u_{i,t}$ is correlated with $\Delta u_{i,t} = u_{i,t} - u_{i,t-1}$ through $u_{i,t}$.

```r
data(Grunfeld, package="plm")
fit.diff = plm(inv ~ capital-1,
               index = c("firm", "year"),
               effect = "individual",
               model = "fd",
               data=Grunfeld)
fit.diff
```

```
Model Formula: inv ~ capital - 1

Coefficients:
capital
0.23078
```

9.4 Within Estimator

An efficient estimator can be obtained by a different transformation. The idea is to consider the individual specific means

$$\overline{Y}_{i\cdot} = \frac{1}{T} \sum_{t=1}^{T} Y_{it}, \quad \overline{\boldsymbol{X}}_{i\cdot} = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{X}_{it}, \quad \overline{u}_{i\cdot} = \frac{1}{T} \sum_{t=1}^{T} u_{it}.$$

Taking the means of both sides of Equation 9.3 implies

$$\overline{Y}_{i\cdot} = \alpha_i + \overline{\boldsymbol{X}}_{i\cdot}'\boldsymbol{\beta} + \overline{u}_{i\cdot}. \tag{9.5}$$

Then, subtracting Equation 9.5 from Equation 9.3 removes the fixed effect α_i from the equation:

$$Y_{it} - \overline{Y}_{i\cdot} = (\boldsymbol{X}_{it} - \overline{\boldsymbol{X}}_{i\cdot})'\boldsymbol{\beta} + (u_{it} - \overline{u}_{i\cdot}).$$

The deviations from the individual specific means are called **within transformations**:

$$\dot{Y}_{it} = Y_{it} - \overline{Y}_{i\cdot}, \quad \dot{\boldsymbol{X}}_{it} = \boldsymbol{X}_{it} - \overline{\boldsymbol{X}}_{i\cdot}, \quad \dot{u}_{it} = u_{it} - \overline{u}_{i\cdot}$$

The within-transfromed model equation is

$$\dot{Y}_{it} = \dot{\boldsymbol{X}}_{it}'\boldsymbol{\beta} + \dot{u}_{it}. \tag{9.6}$$

Hence, to estimate $\boldsymbol{\beta}$, we regress the within-transformed dependent variable $\dot{Y}_{it}$ on the within-transformed regressors $\dot{\boldsymbol{X}}_{it}$ for $i = 1, \ldots, n$ and $t = 1, \ldots, T$.

The within estimator is also called **fixed effects estimator**:

$$\widehat{\boldsymbol{\beta}}_{\text{fe}} = \bigg(\sum_{i=1}^n \sum_{t=1}^T \dot{\boldsymbol{X}}_{it}\dot{\boldsymbol{X}}_{it}'\bigg)^{-1}\bigg(\sum_{i=1}^n \sum_{t=1}^T \dot{\boldsymbol{X}}_{it}\dot{Y}_{it}\bigg).$$

```r
fit.fe = plm(inv ~ capital,
             index = c("firm", "year"),
             effect = "individual",
             model = "within",
             data=Grunfeld)
fit.fe
```

```

Model Formula: inv ~ capital

Coefficients:
capital
0.37075
```

Under (A2-fe), the collection of the within-transformed variables if individual i,

$$(\dot{Y}_{i1}, \ldots, \dot{Y}_{iT}, \dot{\boldsymbol{X}}_{i1}, \ldots, \dot{\boldsymbol{X}}_{iT}, \dot{u}_{i1}, \ldots, \dot{u}_{iT}),$$

forms an i..i.d. sequence for $i = 1, \ldots, n$. The within-transformed variables satisfy (A1-pool)–(A4-pool).

Hence, we can apply the cluster-robust covariance matrix estimator of the pooled regression to the within-transformed variables:

$$\widehat{\boldsymbol{V}}_{\mathrm{fe}} = (\dot{\boldsymbol{X}}'\dot{\boldsymbol{X}})^{-1} \sum_{i=1}^{N} \bigg(\sum_{t=1}^{T} \dot{\boldsymbol{X}}_{it} \hat{u}_{it} \bigg) \bigg(\sum_{t=1}^{T} \dot{\boldsymbol{X}}_{it} \hat{u}_{it} \bigg)' (\dot{\boldsymbol{X}}'\dot{\boldsymbol{X}})^{-1},$$

where $\hat{u}_{it}$ now represents the residuals of $\widehat{\boldsymbol{\beta}}_{\mathrm{fe}}$, and $\dot{\boldsymbol{X}}'\dot{\boldsymbol{X}} = \sum_{i=1}^{N} \sum_{t=1}^{T} \dot{\boldsymbol{X}}_{it} \dot{\boldsymbol{X}}_{it}'$

```r
## cluster-robust covariance matrix
Vfe = vcovHC(fit.fe)
Vfe
```

```
            capital
capital 0.003796144
attr(,"cluster")
[1] "group"
```

```r
## cluster-robust standard error
sqrt(Vfe)
```

```
           capital
capital 0.06161285
attr(,"cluster")
[1] "group"
```

```r
## t-test
coeftest(fit.fe, vcov. = Vfe)
```

```

t test of coefficients:

        Estimate Std. Error t value  Pr(>|t|)
capital 0.370750   0.061613  6.0174 9.018e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

9.5 Time Fixed Effects

While individual-specific fixed effects allow to control for variables that are constant over time but vary across individuals, we can also control for variables that are constant across individuals but vary over time. For example, if new government regulations are introduced at a certain point in time that affect all individuals.

We denote time fixed effects by λ_t. The time effects only regression equation is

$$Y_{it} = \lambda_t + \boldsymbol{X}_{it}'\boldsymbol{\beta} + u_{it}. \tag{9.7}$$

Here, $\boldsymbol{X}_{it}$ does not contain any variable that is the same for all individuals, because these variables are captured by the time fixed effect.

To remove λ_t from the equation, we can subtract time specific means on both sides:

$$Y_{it} - \overline{Y}_{\cdot t} = (\boldsymbol{X}_{it} - \overline{\boldsymbol{X}}_{\cdot t})'\boldsymbol{\beta} + (u_{it} - \overline{u}_{\cdot t}).$$

The time specific means are

$$\overline{Y}_{\cdot t} = \frac{1}{n}\sum_{i=1}^{n} Y_{it}, \quad \overline{\boldsymbol{X}}_{\cdot t} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_{it}, \quad \overline{u}_{\cdot t} = \frac{1}{n}\sum_{i=1}^{n} u_{it}.$$

Hence, we regress $Y_{it} - \overline{Y}_{\cdot t}$ on $\boldsymbol{X}_{it} - \overline{\boldsymbol{X}}_{\cdot t}$ to estimate $\boldsymbol{\beta}$ in Equation 9.7.

```
fit.timefe = plm(inv ~ capital,
             index = c("firm", "year"),
             effect = "time",
             model = "within",
             data=Grunfeld)
fit.timefe
```

```
Model Formula: inv ~ capital

Coefficients:
capital
0.53826
```

9.6 Two-way Fixed Effects

We may include both individual fixed effects and time fixed effects. The two-way fixed effects regression equation is

$$Y_{it} = \alpha_i + \lambda_t + \boldsymbol{X}_{it}'\boldsymbol{\beta} + u_{it}. \tag{9.8}$$

Note that λ_t and α_i capture any variable that is the same for all individuals or is time constant. Therefore, the variables in $\boldsymbol{X}_{it}$ must vary both across individuals and over time.

We can use a combination of the different transformations to remove the fixed effects.

- Individual specific mean:
$$\overline{Y}_{i\cdot} = \alpha_i + \overline{\lambda} + \overline{\boldsymbol{X}}_{i\cdot}'\boldsymbol{\beta} + \overline{u}_{i\cdot},$$
where $\overline{\lambda} = \frac{1}{T}\sum_{t=1}^T \lambda_t$.
- Time specific mean:
$$\overline{Y}_{\cdot t} = \overline{\alpha} + \lambda_t + \overline{\boldsymbol{X}}_{\cdot t}'\boldsymbol{\beta} + \overline{u}_{\cdot t},$$
where $\overline{\alpha} = \frac{1}{n}\sum_{i=1}^n \alpha_i$.
- Total mean:
$$\overline{Y} = \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T Y_{it} = \overline{\alpha} + \overline{\lambda} + \overline{\boldsymbol{X}}'\boldsymbol{\beta} + \overline{u},$$
where $\overline{\boldsymbol{X}} = \frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T \boldsymbol{X}_{it}$ and $\overline{u} = \frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T u_{it}$.

To eliminate the individual and time fixed effects in Equation 9.8, we use the **two-way transformation**:

$$\begin{aligned}
\ddot{Y}_{it} &= Y_{it} - \overline{Y}_{i\cdot} - \overline{Y}_{\cdot t} + \overline{Y} \\
\ddot{\boldsymbol{X}}_{it} &= \boldsymbol{X}_{it} - \overline{\boldsymbol{X}}_{i\cdot} - \overline{\boldsymbol{X}}_{\cdot t} + \overline{\boldsymbol{X}} \\
\ddot{u}_{it} &= u_{it} - \overline{u}_{i\cdot} - \overline{u}_{\cdot t} + \overline{u}.
\end{aligned}$$

Applying the two-way transformation on both sides of Equation 9.8 gives

$$\ddot{Y}_{it} = \ddot{\boldsymbol{X}}_{it}'\boldsymbol{\beta} + \ddot{u}_{it}. \tag{9.9}$$

Hence, we estimate $\boldsymbol{\beta}$ by regressing $\ddot{Y}_{it}$ on $\ddot{\boldsymbol{X}}_{it}$.

```
fit.2wayfe = plm(inv ~ capital,
             index = c("firm", "year"),
             effect = "twoways",
             model = "within",
             data=Grunfeld)
fit.2wayfe
```

```

Model Formula: inv ~ capital

Coefficients:
capital
 0.4138
```

Similarly to the pooled and fixed effects estimator, we can use the cluster-robust covariance matrix estimator and cluster-robust standard errors.

```r
## cluster-robust covariance matrix
V2way = vcovHC(fit.2wayfe)
V2way
```

```
            capital
capital 0.003241852
attr(,"cluster")
[1] "group"
```

```r
## cluster-robust standard error
sqrt(Vfe)
```

```
           capital
capital 0.06161285
attr(,"cluster")
[1] "group"
```

```r
## t-test
coeftest(fit.2wayfe, vcov. = V2way)
```

```

t test of coefficients:

        Estimate Std. Error t value  Pr(>|t|)    
capital 0.413802   0.056937  7.2677 1.268e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

9.7 Comparison of panel models

The fixed effects estimators are asymptotically normal under assumptions (A1-fe)–(A4-fe), and the clustered standard errors are consistent.

```r
fit.pool1 = lm(inv~capital, data=Grunfeld)
fit.pool2 = plm(inv~capital,
            index = c("firm", "year"),
            model = "pooling",
            data=Grunfeld)
```

```r
cluster_se = list(
  sqrt(diag(vcovHC(fit.pool1))),
  sqrt(diag(vcovHC(fit.pool2))),
  sqrt(diag(vcovHC(fit.fe))),
  sqrt(diag(vcovHC(fit.timefe))),
  sqrt(diag(vcovHC(fit.2wayfe)))
)
```

```r
stargazer_output = stargazer(fit.pool1, fit.pool2, fit.fe, fit.timefe, fit.2wayfe,
  se = cluster_se,
  add.lines=list(
    c("Firm FE", "No", "No","Yes","No","Yes"),
    c("Year FE", "No", "No","No","Yes","Yes"),
    c("Clustered SE", "No", "Yes", "Yes", "Yes", "Yes")
  ),
  type="latex",
  omit.stat = "f", df=FALSE,
  dep.var.labels="Gross Investment",
  covariate.labels = "Capital Stock",
  header = FALSE,
  table.placement = "!h")
```

9.8 Dummy variable regression

An alternative way to estimate the fixed effects model is by an OLS regression of Y_{it} on $\boldsymbol{X}_{it}$ and a full set of dummy variables, one for each individual in the sample.

For the time fixed effects model, we include a full set of dummy variables for each time point in the sample, and for the two-way fixed effects model, we include individual and time dummies.

This approach is algebraically equivalent to the within and two-way transformations. The coefficients for the auxiliary dummy variables are usually not reported. The coefficients for `capital` are the same as in the table above:

```r
lm(inv ~ capital + factor(firm), data=Grunfeld)
```

```
Call:
lm(formula = inv ~ capital + factor(firm), data = Grunfeld)
```

Table 9.1

	Dependent variable:				
	Gross Investment				
	OLS		*panel linear*		
	(1)	(2)	(3)	(4)	(5)
Capital Stock	0.477***	0.477***	0.371***	0.538***	0.414***
	(0.078)	(0.126)	(0.062)	(0.153)	(0.057)
Constant	14.236	14.236			
	(19.393)	(28.046)			
Firm FE	No	No	Yes	No	Yes
Year FE	No	No	No	Yes	Yes
Clustered SE	No	Yes	Yes	Yes	Yes
Observations	200	200	200	200	200
R^2	0.439	0.439	0.660	0.429	0.599
Adjusted R^2	0.436	0.436	0.642	0.365	0.530
Residual Std. Error	162.850				
Note:			*p<0.1; **p<0.05; ***p<0.01		

```
Coefficients:
   (Intercept)          capital   factor(firm)2   factor(firm)3   factor(firm)4
      367.6130           0.3707        -66.4553       -413.6821       -326.4410
 factor(firm)5    factor(firm)6   factor(firm)7   factor(firm)8   factor(firm)9
     -486.2784        -350.8656       -436.7832       -356.4725       -436.1703
factor(firm)10
     -366.7313
```

```
lm(inv ~ capital + factor(year), data=Grunfeld)
```

```
Call:
lm(formula = inv ~ capital + factor(year), data = Grunfeld)

Coefficients:
     (Intercept)           capital  factor(year)1936  factor(year)1937
```

```
         39.2068           0.5383          22.4605          27.8993
factor(year)1938 factor(year)1939 factor(year)1940 factor(year)1941
        -36.6889         -42.4012         -11.4293           5.3301
factor(year)1942 factor(year)1943 factor(year)1944 factor(year)1945
        -26.2522         -36.3995         -32.3887         -33.0571
factor(year)1946 factor(year)1947 factor(year)1948 factor(year)1949
         -3.6307         -57.8083         -73.1115        -106.8436
factor(year)1950 factor(year)1951 factor(year)1952 factor(year)1953
       -105.8753         -69.2505         -76.6097         -67.6766
factor(year)1954
       -112.6339
```

```r
lm(inv ~ capital + factor(firm) + factor(year), data=Grunfeld)
```

```
Call:
lm(formula = inv ~ capital + factor(firm) + factor(year), data = Grunfeld)

Coefficients:
     (Intercept)          capital   factor(firm)2    factor(firm)3
        354.9166           0.4138        -51.2329        -402.9933
   factor(firm)4    factor(firm)5   factor(firm)6    factor(firm)7
       -303.7443        -479.3182       -327.4387        -422.4257
   factor(firm)8    factor(firm)9  factor(firm)10 factor(year)1936
       -332.2429        -421.0790       -339.0705          23.9405
factor(year)1937 factor(year)1938 factor(year)1939 factor(year)1940
         32.9483         -27.0935        -30.7979           0.5826
factor(year)1941 factor(year)1942 factor(year)1943 factor(year)1944
         19.5836          -8.6393        -17.5675         -13.7593
factor(year)1945 factor(year)1946 factor(year)1947 factor(year)1948
        -13.5253          17.6985        -27.2407         -37.4300
factor(year)1949 factor(year)1950 factor(year)1951 factor(year)1952
        -66.7623         -63.2855        -23.9098         -23.9138
factor(year)1953 factor(year)1954
         -5.1266         -40.1051
```

9.9 Panel R-squared

We can decompose the total variation into within group variation and between group variation:

$$Y_{it} - \overline{Y} = \underbrace{Y_{it} - \overline{Y}_{i\cdot}}_{\text{within group}} + \underbrace{\overline{Y}_{i\cdot} - \overline{Y}}_{\text{between group}}$$

Two different R squared versions:

- Overall R-squared:

$$R^2_{ov} = 1 - \frac{\sum_{i=1}^n \sum_{t=1}^T \hat{u}_{it}^2}{\sum_{i=1}^n \sum_{t=1}^T (Y_{it} - \overline{Y})^2}$$

  Interpretation: Proportion of total sample variation in Y_{it} explained by the model (the usual R-squared).

- Within R-squared

$$R^2_{wit} = 1 - \frac{\sum_{i=1}^n \sum_{t=1}^T \hat{u}_{it}^2}{\sum_{i=1}^n \sum_{t=1}^T (Y_{it} - \overline{Y}_{i\cdot})^2}$$

  Interpretation: Proportion of sample variation in Y_{it} within the individual units is explained by the model.

For a individual-specific fixed effects regression, consider the two equivalent fixed effects estimators from above:

```
## plm object
fit.fe = plm(inv ~ capital,
             index = c("firm", "year"),
             effect = "individual",
             model = "within",
             data=Grunfeld)
## lm object
fit.fe.lsdv = lm(inv ~ capital + factor(firm), data=Grunfeld)
```

The `summary(object)$r.squared` function applied to the plm object returns the within R-squared, and for the lm object it returns the overall R-squared:

```
## within R-squared
summary(fit.fe)$r.squared
```

```
      rsq    adjrsq
0.6597327 0.6417291
```

```r
## overall R-squared
summary(fit.fe.lsdv)$r.squared
```

```
[1] 0.9184098
```

It is not a big surprise that the fixed effects model explains a lot of the total variation in Y_{it}. The equivalent LSDV model assigns each individual its own dummy variable and therefore, by construction, explains a lot of variation between individuals.

The within R squared is often more insightful because it reflects the model's ability to explain the variation within entities over time.

9.10 R-codes

methods-sec09.R